

Phase-Space Structure for Narrow Planetary Rings

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We address the occurrence of narrow planetary rings under the interaction with shepherds. Our approach is based on a Hamiltonian framework of non-interacting particles where open motion (escape) takes place, and includes the quasi-periodic perturbations of the shepherd's Kepler motion with small and zero eccentricity. We concentrate in the phase-space structure and establish connections with properties like the eccentricity, sharp edges and narrowness of the ring. Within our scattering approach, the organizing centers necessary for the occurrence of the rings are stable periodic orbits, or more generally, stable tori. In the case of eccentric motion of the shepherd, the rings are narrower and display a gap which defines different components of the ring.

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I. INTRODUCTION

Narrow rings have impressed us ever since their observation by Galileo. With the discovery of the rings of Uranus in 1977 by stellar-occultation measurements, the view of the rings as a special feature of Saturn changed completely. This went on with the information

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obtained in particular by the Voyager missions, which provided for the first time a close view into the complexity that takes place in the ring systems. We now know that rings appear in all giant planets. They display different morphologies, sizes, masses and physical properties. Different relevant physical phenomena takes place on them: Surface-wave phenomena, electromagnetic forces acting on micrometric charged dust grains and gravitational confinement due to nearby satellites are some examples. Different aspects of ring structure, dynamics and open problems are reviewed in Refs. [1, 2, 3]. New information is becoming available nowadays by the Cassini mission.

After the discovery of Uranus' rings, which are extremely narrow in comparison to those of Saturn, models including confinement due to nearby moons were introduced. In particular, we mention the shepherding model introduced by Goldreich and Tremaine [4], where two (shepherd) moons around the Uranian rings are proposed to bound and to define the sharp edges of the rings. The mechanism to maintain the ring involves angular momentum transfer between the shepherd moons and the ring particles as well as viscous damping due to interparticle collisions. While the full scenario for shepherding has not been fully understood, the presence of dissipation seems to be essential for the argument, which is in addition formulated assuming that the ring boundary is located at a lower-order resonance (see [3]). In this paper, we propose a purely Hamiltonian formulation of the shepherding mechanism that confines the ring, focusing in particular in the phase-space structure that characterizes it. We do this by assuming, in a first approximation, an independent-particle model, i.e., where the ring particles do not interact among their selves. Within our approach, we succeed in explaining the confinement, on the one hand, and also long standing problems like the narrowness of the ring and its sharp edges.

We choose a rather simple but unrealistic example to illustrate the relevant structures in phase space and understand its implications. We emphasize that the example we define here, which is based on a billiard system with an arbitrary large number of non-interacting particles, is simply a rough cartoon of the real interaction between the particles of the ring and the shepherd moons. We are not proposing by any means that the planetary rings are due to real collisions between the particles of the ring and the shepherds, or that the interactions among particles of the ring are unimportant. Our goal is to point out that, within the independent particle picture considered here, the conclusions we can draw from the phase-space structure are *generic* and therefore of interest in the context of planetary

rings. The reason is that the relevant phase-space structures are independent of the precise details of the system. This is the reason we cautiously refer to it as an *example* or *toy* model rather than a *model* for the (shepherded) narrow rings. The reader should not be misguided by this, which simply avoids the complications of working explicitly with $1/r$ potentials.

The paper is organized as follows: In Sec. II we define the toy model within a Hamiltonian formulation, using an inertial reference frame and, what turns to be more general and convenient, using rotating-pulsating coordinates. We consider only the influence of what it would be interpreted as one shepherd moon for simplicity. In Sec. III we consider the occurrence of rings in the case of circular motion of the shepherd. New results including analytical estimates of the bounds of the ring are presented here. In Sec. IV we consider the case of small but non-zero eccentricity, and show how the arguments carry on in this case. Here, we present indirect evidence for the existence of trapped motion in the system under consideration. Finally, in Sec. V we present our conclusions and some outlook of the work.

II. THE TOY MODEL

Consider the planar motion of a massless point particle off *one* circular hard-disk of radius d which moves on a two-dimensional keplerian periodic orbit (Fig. 1). The potential is therefore zero everywhere except at the position occupied by the disk, where it is infinite. The center of the disk describes an elliptic Kepler orbit with one focus at the origin. This example is a rough cartoon of the restricted three-body problem [5]: In the present case it is defined by the central planet, one shepherd moon orbiting the planet and one particle of the ring. We restrict ourselves here to the three-body version of the problem to simplify the discussion. Results for the four-body case will be briefly discussed in Sec. V.

The radial position of the center is denoted by $R(\phi)$, where ϕ defines the position along the Kepler orbit measured from the pericenter. $R(\phi)$ is given by the usual expression in polar coordinates in terms of the semi-major axis $a = 1$ and the eccentricity ε . We shall focus on the case of small or zero eccentricity, since this is the situation commonly encountered. For non-zero ε , due to the explicit time dependence the system has two-and-half degrees of freedom and no constant of motion. Hence, the phase space is five dimensional: Two coordinates and their canonically conjugated momenta define the phase-space coordinates of the particle, and the angle ϕ determines the position of the disk along the Kepler orbit.

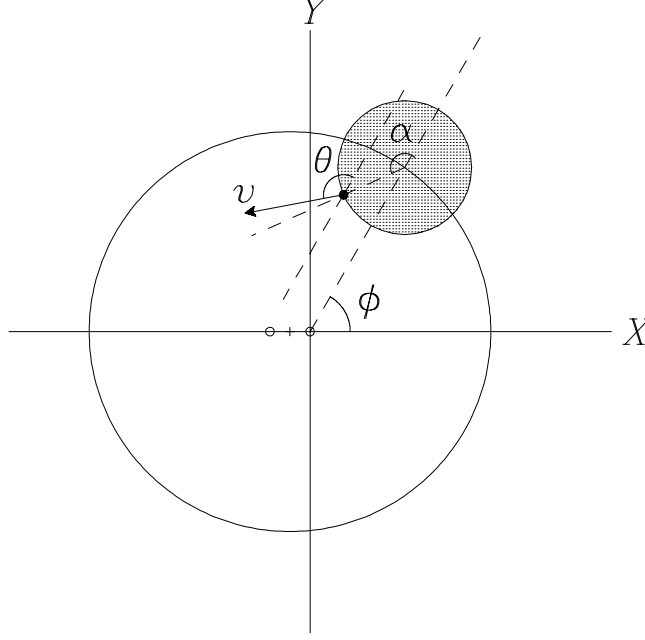


FIG. 1: Geometry of the toy model: ϕ denotes the initial position of the center of the disk; α denotes the position of collision point on the disk; v denotes the magnitude of the outgoing velocity and θ defines its direction. The foci of the ellipse described by the center of the disk are shown on the X -axis as open circles (\circ), and the center by a cross ($+$).

In the circular case, as we show below, there is a constant of motion which reduces the dimensionality of phase space.

The Hamiltonian, expressed in an inertial reference frame, can be written as

$$H = \frac{P_X^2 + P_Y^2}{2} + V(|\vec{X} - \vec{X}_d(\phi)|). \quad (1)$$

Here, \vec{X}_d denotes the position vector of the center of the disk, which implies an explicit time-dependent potential through the dependence upon ϕ , and \vec{X} is the position of the particle. The potential $V(|\vec{X} - \vec{X}_d(\phi)|)$ is zero for $|\vec{X} - \vec{X}_d(\phi)|^2 > d^2$ and infinite otherwise. We perform a canonical transformation to rotating-pulsating coordinates by means of the generating function

$$W = \frac{R(\phi)}{\bar{R}} [P_X(x \cos \phi - y \sin \phi) + P_Y(x \sin \phi + y \cos \phi)],$$

where \bar{R} is the mean-orbital radius of the disk. In this coordinates, the disk is at rest at $\vec{x}_d = (\bar{R}, 0)$. The new Hamiltonian is then given by

$$J = \frac{1}{2} \frac{\bar{R}^2}{[R(\phi)]^2} (p_x^2 + p_y^2) - \dot{\phi} (xp_y - yp_x) - \dot{\phi} \frac{1}{R(\phi)} \frac{dR(\phi)}{d\phi} (xp_x + yp_y) + V(|\vec{x} - \vec{x}_d|). \quad (2)$$

Note that in the case of a circular orbit we have $R = a$ and $\phi = \omega t$, where $\omega = \dot{\phi}$ is the orbital frequency and t is the time. In this case, the third term of the r.h.s. of the Hamiltonian (2) vanishes. By consequence, J is time independent, and thus a constant of motion. This is the so-called Jacobi integral of the restricted three-body problem. We shall refer to the value of J in Eq. (2) as the Jacobi integral, even in the case of non-vanishing eccentricity. In the following we shall drop the potential in Eq. (1) and (2), and restrict implicitly to the case $|\vec{X} - \vec{X}_d(\phi)|^2 = (R(\phi)/\bar{R})^2 |\vec{x} - \vec{x}_d|^2 > d^2$.

The dynamics of this scattering system are straightforward. The particle moves freely on a rectilinear trajectory of constant velocity until it encounters the disk. No collision leads to open motion, i.e., the particle escapes away from the interaction region. In the context of the ring, this situation is interpreted as the case of a particle abandoning the bulk of the ring. If a collision takes place, the particle is specularly reflected with respect to the local (moving) frame of the disk at the collision point. This defines the outgoing conditions after the collision, and then the motion is again rectilinear uniform [5]. The precise result of a collision depends on the position where the collision occurs on the disk, the relative velocities, and for non-vanishing ε , on the position along the elliptic trajectory of the disk. Collisions taking place on the front of the disk increase the (outgoing) kinetic energy of the particle, while collisions on the back reduce it.

In the context of the rings, we are interested in (ring) particles which are dynamically trapped; within the present example, this can only be through consecutive collisions with the disk. A convenient description of this situation is given by introducing the following quantities, which are defined at the collision point with the disk (cf. Fig. 1). The angle ϕ characterizes the position of the center of the disk with respect to the X axis (inertial frame), and v is the magnitude of the velocity after the collision (outgoing velocity). The angle α denotes the angle formed by $\vec{X}_d(\phi)$ and the position vector of the collision point referred to the center of the disk. Finally, the angle θ defines the outgoing direction of the velocity.

III. OCCURRENCE OF RINGS: THE CIRCULAR CASE

The dynamics of the disk define naturally a map at the collision point with the disk. This map is *open* in the sense that certain initial conditions may not have an associated image.

This situation corresponds to the case where the particle escapes after a given collision. On the other hand, trapped trajectories are associated with consecutive collisions with the disk, irrespective whether such trajectories display periodic, quasi-periodic or chaotic behavior. Periodic orbits and their stability are thus of central interest in the context of the rings. In particular, simple periodic orbits for the circular case can be worked out analytically for the example considered here.

Between collisions, i.e. during the free motion of the particle, the linear equations of motion in the rotating frame can be solved explicitly. The Jacobi integral is then given by ($\omega = 1$)

$$J = \frac{v^2}{2} - v(R \sin \theta + d \sin(\theta - \alpha)). \quad (3)$$

We notice that the radial collisions ($\alpha = \pi$) do conserve the kinetic energy, and are thus fixed points of the dynamical map in the rotating frame. Starting from the initial conditions of a radial collision, in order to have that the next collision is radial too the velocity must be given by

$$v = -\frac{2(R-d) \cos \theta}{\Delta \phi} = -\frac{2(R-d) \cos \theta}{(2n-1)\pi + 2\theta}. \quad (4)$$

Here, $n = 0, 1, \dots$ denotes the number of full turns completed by the disk between consecutive radial collisions, and $\Delta \phi = (2n-1)\pi + 2\theta$ is the corresponding change in the angle ϕ . For these orbits, the Jacobi integral is given by

$$J_n = (R-d)^2 \frac{2 \cos^2 \theta + \Delta \phi \sin(2\theta)}{(\Delta \phi)^2} = (R-d)^2 \frac{2 \cos^2 \theta (1 + \Delta \phi \tan \theta)}{(\Delta \phi)^2}. \quad (5)$$

In Fig. 2a we illustrate the structure of $J_n/(R-d)^2$, for some values of n . Note that the curves display one maximum and one minimum for each value of n . Moreover, for negative values of J , two periodic orbits can have the same values of θ and the Jacobi integral, while the characteristic n 's are different. This situation is a consequence of the quadratic character of Eq. (3), which for $J < 0$ has two distinct solutions for v . The occurrence of maxima and minima on these curves suggests the appearance of these consecutive collision orbits by saddle-center bifurcations.

For $\varepsilon = 0$, Eq. (2) defines a time-independent two degrees of freedom system. In this case, the stability of the periodic orbits can be checked by standard procedures involving the trace and the determinant of the linearized matrix of the Poincaré map at the fixed points. The transformation $(\alpha_{n+1}, p_{n+1}) = \mathcal{P}_J(\alpha_n, p_n)$ with

$$p_n = -d - R \cos \alpha_n - v \sin(\alpha_n - \theta_n), \quad (6)$$

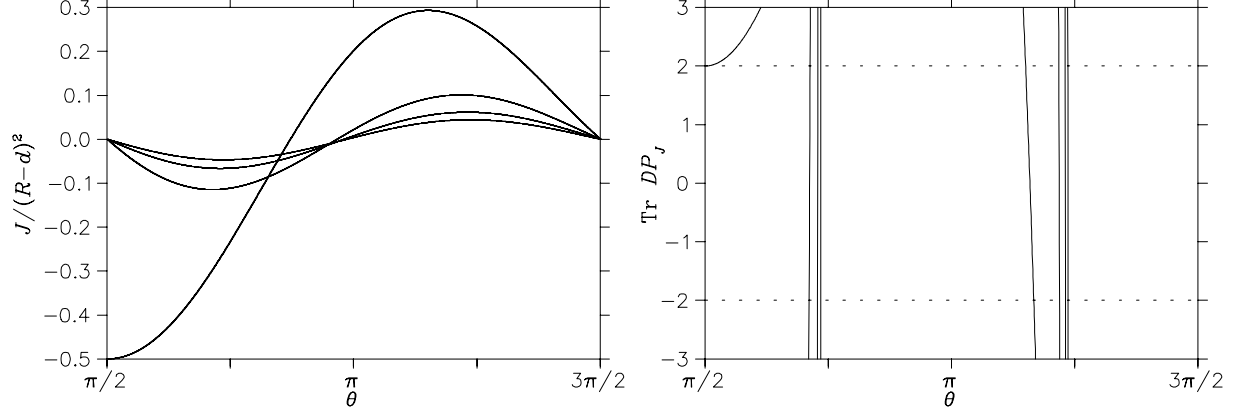


FIG. 2: Consecutive radial collision orbits (simple periodic orbits) of the circular rotating billiard for $n = 0, 1, 2, 3$: (a) Jacobi integral J in units of $(R - d)^2$ and (b) $\text{Tr } DP_J$ in terms of θ .

defines a canonical Poincaré map. Here, $\alpha_n \in [0, 2\pi]$ and $p_n \in [-p_{\max}, p_{\max}]$, where $p_{\max} = (2J + R^2 + d^2 + 2Rd \cos \alpha)^{1/2}$. The determinant of the linear approximation is thus 1. [The same map but with $p_n = \theta$ which was used in Ref. [5] has also unit determinant for the consecutive collision periodic orbits, even-though it is non-canonical.] The information on the stability is thus completely contained in the trace of the linearized matrix DP_J . One finds [6]

$$\text{Tr } DP_J = 2 + \frac{(\Delta\phi)^2(1 - \tan^2 \theta) - 4(1 + \Delta\phi \tan \theta)}{d/R}. \quad (7)$$

It is well known that changes in the stability of the periodic orbits (bifurcations) take place when the characteristic multipliers are ± 1 ; this is equivalent to have $\text{Tr } DP_J = \pm 2$. Equating the r.h.s. of Eq. (7) to 2 turns to be equivalent to the condition $dJ_n/d\theta = 0$ [7], i.e., the condition defining the position of the minima and maxima of J_n . The corresponding condition obtained for -2 yields a condition related to period-doubling bifurcations; we shall come back to this later. Note that $\text{Tr } DP_J = 2$ is independent of d/R , while in the case $\text{Tr } DP_J = -2$ there is an explicit dependence on d/R . Figure 2b shows the behavior of $\text{Tr } DP_J$ in terms of θ .

We observe from Fig. 2b that there are connected intervals in θ , and therefore in J , where radial collision periodic orbits are strictly stable, i.e. $2 > \text{Tr } DP_J > -2$. For J within these intervals of Jacobi integral, the phase-space structure of the scattering system displays one elliptic fixed point, surrounded by the typical KAM tori and some chaotic layers (Fig. 3a). This region in phase space is bounded by the stable and unstable invariant manifolds of the companion hyperbolic fixed point, which reach the incoming and outgoing asymptotes of

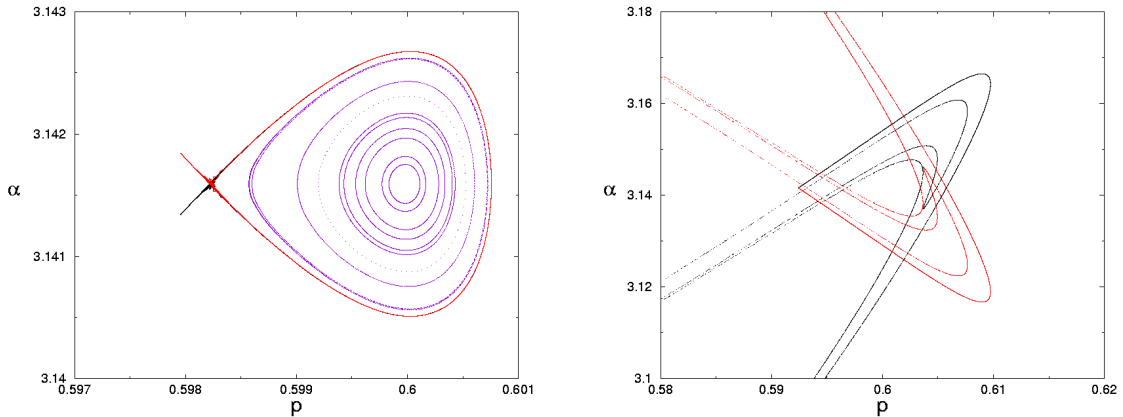


FIG. 3: Phase-space structure of the scattering billiard on a circular orbit ($R = 1$ and $d = 1/2$). (a) Situation with one elliptic fixed point ($J/(R-d)^2 = 0.29325$). (b) Case in which the elliptic fixed point has turned into inverse hyperbolic ($J/(R-d)^2 = 0.29218$). The phase space is dominated by unstable motion, although the Smale horseshoe is not complete.

the scattering system. The motion of particles whose initial conditions lie within this region display consecutive collisions and therefore are dynamically trapped. We shall refer to this region as the region of trapped motion. Note that if the initial conditions of the particle do not belong to any region of trapped motion, the particle will eventually escape from the interaction region.

We consider the case $J > 0$ for concreteness. By further reducing the value of the Jacobi integral, there is a value where $\text{Tr } D\mathcal{P}_J = -2$ is fulfilled. There, the elliptic fixed point becomes inverse hyperbolic, and a period doubling bifurcation sets in: The region of trapped motion vanishes rapidly after further reducing J . In Fig. 3b we have illustrated this case plotting the phase-space structure corresponding to an incomplete Smale horseshoe [8]. We remark that the dependence upon d/R of $\text{Tr } D\mathcal{P}_J = -2$ implies that the actual parameters of the billiard define the thickness of the ring. Physically this implies that the parameters related with the motion and mass of the shepherd moons do indeed influence the width of the ring. Further reducing the value of J yields a hyperbolic Smale horseshoe. A detailed description of the complete bifurcation scenario for this billiard, as well as the dynamics for $J = 0$, can be found in Ref. [6].

We turn now to the rings [7]. Within the present toy model, consider an arbitrary large number of *non-interacting* particles. The initial conditions of these particles are completely

arbitrary, i.e. have no restrictions at all (e.g., in the value of the Jacobi integral). Then, the particles may collide with the disk at any value ϕ along the circular orbit of the disk. We are thus considering the most general ensemble of initial conditions. Letting the system evolve most particles escape after few collisions. Yet, those whose initial conditions belong to the phase-space regions which are trapped by consecutive collisions with the disk will not escape. This situation may correspond to the initial conditions of the unstable periodic orbits, or to homoclinic or heteroclinic orbits. These initial conditions are of no interest for us since they are a set of measure zero. On the other hand, initial conditions corresponding to values of J with a phase-space structure like in Fig. 3a, have in addition the important property of having a small but finite, i.e. non-zero, measure.

We focus on the latter set of initial conditions in phase space, which we refer to as the particles of the ring, or ring particles. The motion of the ring particles, as mentioned before, may be periodic, quasi-periodic or chaotic. Therefore, we are not assuming or imposing any specific relation with the period of the disk, $2\pi/\omega$, other than to assure consecutive collisions. This is important because no resonance condition is implied: Even in the case of periodic motion, the precise period of the radial collision periodic orbits, $\Delta\phi/\omega$, may not be a rational of the period of the disk. Of particular interest is certainly the pattern formed by this set of particles in X - Y space (inertial frame). Figures 4 show the pattern obtained after few thousands of revolutions of the disk, which results from a large number of particles whose initial conditions are close to the maximum of the $J_{n=0}$ curve. The plot shows the (X, Y) position of all particles, in an inertial frame, that have not escaped at the time when the “photograph” was taken. Rigorously speaking, there may be still some particles in this plot which do not belong to the region of trapped motion although they have not escaped when the plot was made. We shall simply say that the number of such particles is much smaller than the number of particles that stay in the ring for ever. The plot justifies by itself to call this pattern a *ring*.

In Fig. 4 we also plot the curves corresponding to $\text{Tr } D\mathcal{P}_J = \pm 2$. These curves approximate the boundaries of the ring, which follows directly from the stability arguments used for its construction. They do not define true boundaries of the ring since they are related with the appearance and destruction of the central elliptic fixed point; the satellite structure around them defines some further thickness. Note that the nature of these curves, i.e. the whole bifurcation scenario, implies naturally sharp edges of the ring. It is also worth

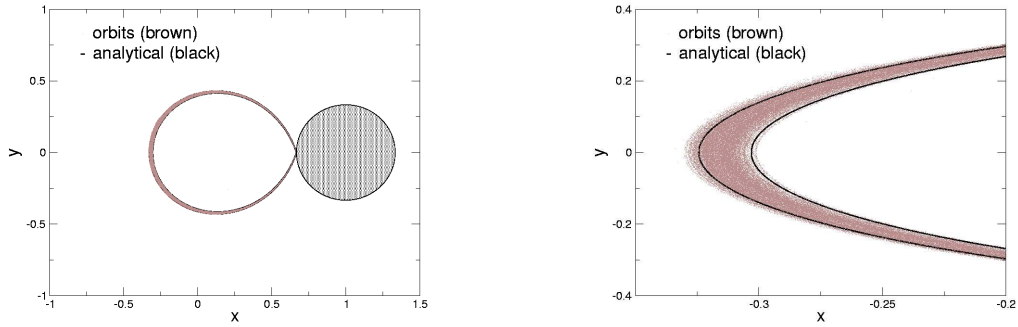


FIG. 4: (a) Stable ring of non-interacting particles associated with the $n = 0$ stable region of the disk on a circular orbit. (b) Zoom of a region of the ring. The red lines represent the limits defined by $\text{Tr } DP_J = \pm 2$.

noticing the fact that the narrowness of the ring is a consequence of the relatively small area occupied by the region of trapped motion in phase space; this also takes into account the (small) variations induced by changing J . Furthermore, the ring can be characterized as eccentric, in the sense that there are two points, the periapse and apoapse, forming a line that intersects the origin, the line of apsides. This property is collective, in the sense that it is defined by the ensemble of particles that builds the ring. We emphasize that these properties do not depend on the particular interaction we have considered here: They hold for any scattering system which displays stable motion confined to a tiny region in phase space. This implies that the arguments hold under generic small perturbations. Putting it differently, the phase-space structures in which we are interested do display structural stability. The robustness of the argument follows from the fact that the rings are constructed from phase-space considerations which hold generically.

Further properties of the ring, which may not be generic as the preceding ones, are the following: For each situation in which the consecutive radial collision orbits are stable, one such ring will be found. In the context of the present example, this is so around each maximum and minimum of the curves J_n . The ring particles move along the same direction of the disk for negative J , while the motion is retrograde for $J > 0$. The motion of the ring is a rigid rotation around the origin, having the same period of the disk. All rings share the same line of apsides. Finally, while it holds generically that the rings are eccentric, the rather large eccentricity found in the present example is surely an artifact of the toy model.

We shall finish this section commenting that our present approach only serves to describe the occurrence of rings. We cannot distinguish whether the rings appeared at the same time that the main planet and the shepherds, or whether they formed after a catastrophic event like a collision with a big body. In the first case, the ring particles are spread all over in phase space, and those which remain within the ring are the ones within the regions of trapped motion. We can consider the second option like a swarm of particles whose initial conditions differ little and are defined on a small but closed region in phase space, in particular in a small neighborhood in position space. Again, the particles that remain in the ring are within the region of trapped motion. However, they do not initially fill all this region: Time evolution makes that they spread over the region of trapped motion in a short time-scale and finally define a ring. This situation is similar to thermalization processes in standard statistical mechanics.

IV. RINGS IN THE ELLIPTIC CASE

Here, we shall show that the same line of argumentation and the construction described above can be carried on when the disk is moving on a Kepler ellipse. We must emphasize that this is not a trivial generalization. We remark that the Hamiltonian (2) can be interpreted as a two degrees of freedom system with the addition of a quasi-periodic time-dependent perturbation. Then, the system has effectively two-and-half degrees of freedom and the Jacobi integral in the rotating-pulsating coordinates is no longer a conserved quantity. Therefore, phase space is five dimensional and Poincaré surfaces of section are of no practical use. Recent research on chaotic scattering in many dimensions has revealed new interesting and intricate behavior [9, 10]. Furthermore, and more important, the argument used to prove the existence of stable motion in the circular case relies on the *generic* character of the saddle-center bifurcation. This is only valid when a *single* parameter is varied; for the circular case this is a consequence of the conservation of the Jacobi integral. However, for non-zero eccentricity there is no constant of motion, and hence, the bifurcation scenario involves at least two parameters.

As mentioned above, the elliptic motion introduces a kind of quasi-periodic time perturbation on the circular case. It is therefore meaningless to look for periodic motion, as we did in the circular case. We remark though that the analytical continuation method [11] can be

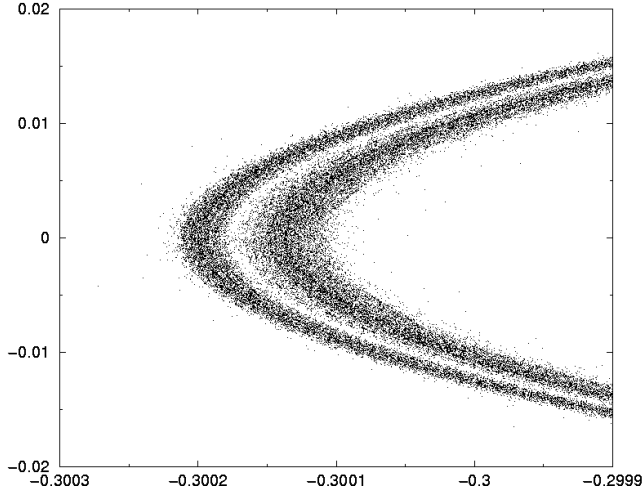


FIG. 5: Detail of the ring of non-interacting particles when the disk moves on an eccentric Kepler orbit ($\varepsilon = 0.00165$). Notice the low density of particles found at the middle region of the ring.

used to prove the existence of periodic orbits; these are deformations from a class of radial collision orbits of the circular case [12]. The important organizing centers in phase space are in this case stable tori [13]. While more complicated situations may arise with respect to the stability of such invariant structures in phase space, due to its larger dimensionality, we shall mention that stable tori can indeed be found, at least for quite small values of the eccentricity. Below we present evidence that supports this. Close to these stable tori, the structure of phase space is somewhat similar to the one discussed for the circular motion. Then, there is a region of dynamical trapped motion due to consecutive collisions. The motion of non-interacting particles started there can be quasi-periodic or chaotic. Hence rings occur in the same way: Plotting at a given time the position of the ensemble of non-interacting particles that are not ejected after a few thousand collisions yields a ring, as shown in Fig. 5. We notice that the ring is also sharp-edged, eccentric and narrow: As mentioned above, these properties hold generically. We remark that the ring in this case is narrower than in the circular case. This may be interpreted as some destabilizing effect due to the eccentricity of the shepherd moon. In Fig. 5 we also observe a gap, or region of comparatively low density of particles, which divides the ring in two components. We have no satisfactory explanation for the occurrence of this region at the moment. While it is tempting to associate this gap with different strands or ring components as those observed in Saturn's F ring, and its occurrence to the eccentricity of the shepherds, more understanding on this aspect is certainly needed.

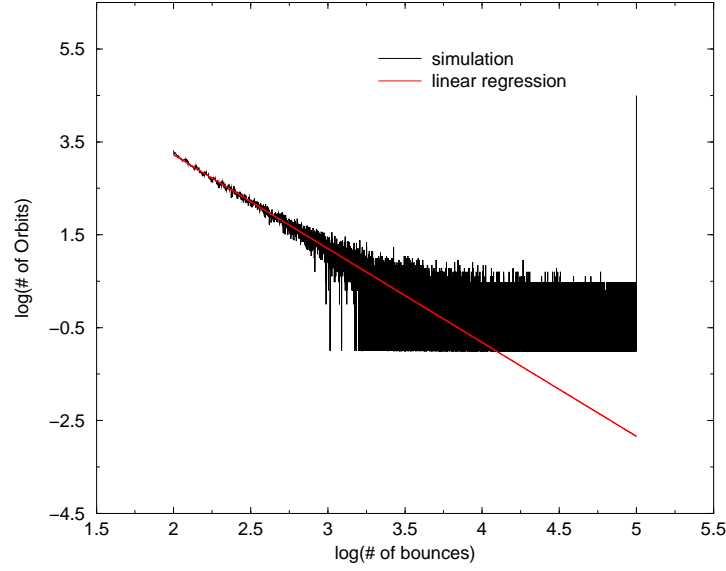


FIG. 6: Differential escape rate for $\varepsilon = 0.00165$. The red line shows the results obtained by a linear regression. The peak at the right of the plot indicates the existence of particles that do not escape with an algebraic decay. These particles are dynamically trapped.

We present now some evidence that pure stable quasi-periodic tori indeed exist. First, we mention that we have found trajectories that survive more than 100000 collisions. While this number of bounces is quite large in the numerical calculations, these trajectories may still not be trapped due to the large dimensionality of phase space, in particular if they display chaotic motion. Indeed, Arnold diffusion [14, 15] could be relevant in the present situation. Yet, the typical time scale in which Arnold diffusion takes place is exponentially large and thus our simulations have not detected it. Secondly, we investigate the escape rate of an ensemble of initial conditions near a region where we suspect there is trapped motion. The escape rate of a hyperbolic system shows an exponential decay in contrast to the algebraic one observed for a non-hyperbolic system [16]. For the latter, the exponent is close to 2. We thus expect to have an algebraic decay, with non-statistical deviations on the tail if there are initial conditions within a region of trapped motion. In Fig. 6 the differential escape rate is shown for $\varepsilon = 0.00165$, showing the expected (initial) algebraic decay with exponent ≈ 2.2 . The large peak that appears in the tail at the cut-off in the number of bounces indicates that there is non-zero overlap between the region where the initial conditions were chosen, and the region of stable trapped motion associated with a central stable tori. Therefore, a stable trapped motion does exist for this eccentricity, which

yields the ring shown in Fig. 5. Notice that this method, while it does not provide any information of the size in phase space of the trapped region or on dynamics, shows that the measure of the set is strictly non-zero.

V. CONCLUSIONS AND OUTLOOK

In this paper we have discussed generic mechanisms in phase space for the occurrence of narrow eccentric planetary-like rings due to the interaction with shepherd moons. Our approach is based on a Hamiltonian formulation and assumes non-interacting particles. The non-interacting character of our approach is only a first approximation in the same sense that the mean-field theory is for nuclear physics. We have used as a simple but unrealistic example for the argumentation: An open (scattering) billiard system on a Kepler orbit. We have focused on the structure of phase space, and extended previous results to the case of an eccentric Kepler orbit. Although the general arguments carry over, the generalization is not straightforward: The enlarged dimensionality introduces new possibilities in the stability properties of the fixed points and on the bifurcation scenario involving now at least two parameters. We find that the eccentric character of the orbit indeed influences the structure of the ring, which turns out to be narrower in comparison to the circular counterpart and also displays a gap. We have no satisfactory explanation for the latter aspect yet, and do not know whether it is an artifact of our model example or something with deeper significance. We have also provided a more detailed analytical framework to study the circular case, in particular providing estimates for the width of the ring. We emphasize that these new results are by no means a trivial extension: The dimensionality of the system implied through the time-dependence makes this a subtle task. To find rings in the present toy example which has no attractive interaction shows the genericity of the arguments.

The relevant structures in phase space correspond to those of pure trapped motion, which are organized around stable periodic orbits, or more generally, around stable tori. In phase space these regions are bounded by the invariant manifolds of unstable orbits. Within these regions no escape is possible. If there exists such a set of pure bounded motion in phase space, it is strictly of non-zero measure. An ensemble of (non-interacting) particles defined by their initial conditions within these regions yields in position space a ring. The motion of particles with initial conditions in these regions must be contrasted with those outside:

In the latter case, the particles are ejected from the system after few periods of rotation. This situation leads naturally to sharp-edges of the ring. This scenario is *robust*, in the sense that it does not depend on the particular interaction between the particles of the ring, the shepherd moons and the central planet: The rotation of the shepherd moon by itself can build the conditions to have trapped motion. Once there is pure trapped motion the existence of such a ring follows immediately. That is, within a more realistic description, non-keplerian effects like planet oblateness can also be accounted for. The rings obtained display sharp-edges, are eccentric and narrow. We emphasize that these properties do not assume any specific resonance condition between the particle motion and the shepherd moons, as it is often considered or assumed. It is the existence of stable motion and its organizing centers in phase space what determines the existence of such a ring. We believe that this scenario may provide an explanation to the observed feature of apse alignment, which remains an open problem within the usual shepherding model.

This example can be taken over to consider the interaction with two disks on non-overlapping keplerian orbits, in order to model the more realistic situation of two shepherd moons. Using the same ideas, it is possible to define a (geometrical) criterion essentially given by the intersection of the ring (due to the outer disk) and the orbit of the inner one. This criterion may display a specific dependence on the actual position of the outer ring in the case of eccentric orbits. A possible generalization to the case with smooth potentials and in particular the $1/r$ case [17], should maybe incorporate the results related to the stability of the organizing center, and then provide estimates of the width of the ring.

Interparticle interactions do indeed play a crucial role in the dynamics of narrow rings. Our current point of view is that they can be incorporated within the present modeling, at least for a dilute ring, as some chaotic diffusion (Brownian motion) in phase space, with probably some constraints. Such diffusive process induces small fluctuations in the value of the Jacobi integral and in the instantaneous direction of motion of the particles. One may thus impose that such a collision process conserves the total angular momentum of the colliding particles. This will effectively make that the dimension of phase space incorporates the number of particles. Future research will be carried along this direction.

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